

# ESTIMATES OF ESSENTIAL NORMS OF WEIGHTED COMPOSITION OPERATOR FROM BLOCH TYPE SPACES TO ZYGmund TYPE SPACES

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**ABSTRACT.** Let  $u$  be a holomorphic function and  $\varphi$  a holomorphic self-map of the open unit disk  $\mathbb{D}$  in the complex plane. We give some new characterizations for the boundedness of the weighted composition operators  $uC_\varphi$  from Bloch type spaces to Zygmund type spaces in  $\mathbb{D}$  in terms of  $u, \varphi$ , their derivatives and the  $n$ -th power  $\varphi^n$  of  $\varphi$ . Moreover, we obtain some similar estimates for their essential norms. From which the sufficient and necessary conditions of compactness of the operators  $uC_\varphi$  follows immediately.

## 1. INTRODUCTION

Let  $\mathbb{D}$  be the open unit disk in the complex plane  $\mathbb{C}$ . Let  $H(\mathbb{D})$  denote the class of all functions analytic on  $\mathbb{D}$  and  $S(\mathbb{D})$  the collections of all holomorphic self-map of  $\mathbb{D}$ . We give the weighted Banach spaces of analytic functions

$$H_\nu^\infty = \{f \in H(\mathbb{D}) : \|f\|_\nu := \sup_{z \in \mathbb{D}} \nu(z)|f(z)| < \infty\}$$

endowed with norm  $\|\cdot\|_\nu$ , where the weight  $\nu : \mathbb{D} \rightarrow \mathbb{R}_+$  is a continuous strictly positive and bounded function. The weight  $\nu$  is called radial, if  $\nu(z) = \nu(|z|)$  for all  $z \in \mathbb{D}$ . For a weight  $\nu$  the associated weight  $\tilde{\nu}$  is defined by

$$\tilde{\nu}(z) := (\sup\{|f(z)|; f \in H_\nu^\infty, \|f\|_\nu \leq 1\})^{-1}, \quad z \in \mathbb{D}.$$

It is obvious that  $\tilde{\nu}_\alpha = \nu_\alpha$  for the standard weights  $\nu_\alpha(z) = (1 - |z|^2)^\alpha$ , where  $0 < \alpha < \infty$ . Besides the standard weights  $\nu_\alpha$ , we also consider the logarithmic weight

$$\nu_{\log}(z) := \left( \log \left( \frac{e}{1 - |z|^2} \right) \right)^{-1}, \quad z \in \mathbb{D}.$$

It is not difficult to see that also  $\tilde{\nu}_{\log} = \nu_{\log}$ . Moreover, the Banach space of bounded analytic functions on  $\mathbb{D}$  is denoted by  $H^\infty$ . In the following, let  $\|f\|_{\nu_\alpha}$  and  $\|f\|_{\nu_{\log}}$  denote the norms defined on the weighted Banach spaces  $H_{\nu_\alpha}^\infty$  and  $H_{\nu_{\log}}^\infty$ .

Recall that the Bloch type space  $\mathcal{B}^\alpha$  on the unit disk, consists of all  $f \in H(\mathbb{D})$  satisfying

$$\|f\|_\alpha := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty$$

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endowed with the norm  $\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \|f\|_\alpha < \infty$ . As we all known that for  $0 < \alpha < 1$ ,  $\mathcal{B}^\alpha$  is a subspace of  $H^\infty$ . When  $\alpha = 1$ , we get the classical Bloch space  $\mathcal{B}$ .

For  $0 < \beta < \infty$ , we denote by  $\mathcal{Z}_\beta$  the Zygmund type space of those functions  $f \in H(\mathbb{D})$  such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |f''(z)| < \infty$$

equipped with the norm

$$\|f\|_{\mathcal{Z}_\beta} := |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |f''(z)|.$$

For  $\beta = 1$  we obtain the classical Zygmund space  $\mathcal{Z}$ .

The composition operator  $C_\varphi$  induced by  $\varphi \in S(\mathbb{D})$  is defined on  $H(\mathbb{D})$  by  $C_\varphi(f) = f \circ \varphi$  for any  $f \in H(\mathbb{D})$ . This operator is well studied for many years, we refer to the books [1, 13], which are excellent sources for the development of the theory of composition operators in function spaces. For  $u \in H(\mathbb{D})$ , we define the weighted composition operator

$$uC_\varphi f(z) = u(z)f(\varphi(z)), \text{ for } f \in H(\mathbb{D}).$$

It is obvious that  $uC_\varphi = C_\varphi$  when  $u$  is the identity map.

The essential norm of a continuous linear operator  $T$  is the distance from  $T$  to the compact operators  $K$ , that is  $\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}$ . Notice that  $\|T\|_e = 0$  if and only if  $T$  is compact, so estimates on  $\|T\|_e$  lead to conditions for  $T$  to be compact. There are lots of papers concerning this topic, the interested readers can refer to [3, 4, 10, 14, 19, 20] and the references therein.

Recently, there is an increase interest to characterize the boundedness and compactness of composition operators acting on Bloch type spaces in terms of the  $n$ -th power  $\varphi^n$  of  $\varphi$ , see [7, 8, 16, 17]. The similar characterization between Bloch-type spaces with general radial weights was obtained by Hyvärinen et al in [5]. The natural question to ask is whether the essential norm formula for composition operators between Bloch-types paces  $\mathcal{B}_\alpha$  can be generalized to weighted composition operators. In 2012, Manhas and Zhao [11] showed that the question has an affirmative answer when  $\alpha \neq 1$ ; however, they were not able to estimate the essential norm of weighted composition operators on the Bloch space  $\mathcal{B}$ . The open problem was solved by Hyvärinen and Lindström in [6]. Moreover, they presented a direct method to calculate the essential norms of weighted composition operators  $uC_\varphi$  acting on all Bloch-type spaces  $\mathcal{B}_\alpha$  in terms of  $u$  and the  $n$ -th power of  $\varphi$ . After that, Esmaeili and Lindström [2] gave similar characterizations for the weighted composition operators acting on Zygmund type spaces.

Based on the above foundations, in this paper we used an approach due to Hyvärinen and Lindström in [6] and Esmaeili and Lindström in [2] to obtain new characterizations for bounded weighted composition operators from Bloch type spaces to Zygmund type spaces, and to give similar estimates of the essential norms of such operators.

Throughout the remainder of this paper,  $C$  will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation  $A \preceq B$ ,  $A \succeq B$  and  $A \asymp B$  mean that there may be different positive constants  $C$  such that  $A \leq CB$ ,  $A \geq CB$  and  $B/C \leq A \leq CB$ .

## 2. SOME LEMMAS

In this section, we give some auxiliary results which will be used in proving the main results of the paper. The following two lemmas is crucial to the new characterizations.

**Lemma 2.1.** [9, Theorem 2.1] or [5, Theorem 2.4] *Let  $\nu$  and  $w$  be radial, non-increasing weights tending to zero at the boundary of  $\mathbb{D}$ . Then*

*(i) the weighted composition operator  $uC_\varphi$  maps  $H_\nu^\infty$  into  $H_w^\infty$  if and only if*

$$\sup_{n \geq 0} \frac{\|u\varphi^n\|_w}{\|z^n\|_\nu} \asymp \sup_{z \in \mathbb{D}} \frac{w(z)}{\tilde{\nu}(\varphi(z))} |u(z)| < \infty,$$

*with norm comparable to the above supremum.*

$$(ii) \|uC_\varphi\|_{e, H_\nu^\infty \rightarrow H_w^\infty} = \limsup_{n \rightarrow \infty} \frac{\|u\varphi^n\|_w}{\|z^n\|_\nu} = \limsup_{|\varphi(z)| \rightarrow 1} \frac{w(z)}{\tilde{\nu}(\varphi(z))} |u(z)|.$$

**Lemma 2.2.** [6, Lemma 2.1] *For  $0 < \alpha < \infty$  we have*

$$(i) \lim_{n \rightarrow \infty} (n+1)^\alpha \|z^n\|_{\nu_\alpha} = \left(\frac{2\alpha}{e}\right)^\alpha.$$

$$(ii) \lim_{n \rightarrow \infty} (\log n) \|z^n\|_{\nu_{\log}} = 1.$$

The next lemma is a well-known characterization for the Bloch-type space on the unit disc, see [18].

**Lemma 2.3.** *For  $f \in H(\mathbb{D})$ ,  $m \in \mathbb{N}$  and  $\alpha > 0$ , then*

$$f(z) \in \mathcal{B}^\alpha \Leftrightarrow \|f\|_\alpha \asymp \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha+m-1} |f^{(m)}(z)| < \infty.$$

Hence, when  $f \in \mathcal{B}^\alpha$ , we have that

$$\|f\|_\alpha \asymp \sup_{z \in \mathbb{D}} (1 - |z|)^{\alpha+m-1} |f^{(m)}(z)| < \infty. \quad (2.1)$$

where  $f^{(m)}$  denotes the  $m$ -th order derivative of  $f \in H(\mathbb{D})$ .

**Lemma 2.4.** [12, 18] *For  $\alpha > 0$ ,  $f \in \mathcal{B}^\alpha$ , then we have that*

$$|f(z)| \leq C \begin{cases} \|f\|_{\mathcal{B}^\alpha}, & 0 < \alpha < 1; \\ \|f\|_{\mathcal{B}^\alpha} \log \frac{e}{1-|z|^2}, & \alpha = 1; \\ \frac{1}{(1-|z|^2)^{\alpha-1}} \|f\|_{\mathcal{B}^\alpha}, & \alpha > 1. \end{cases}$$

*for some  $C$  independent of  $f$ .*

The following lemma is a special case of [15, Lemma 6].

**Lemma 2.5.** *For  $0 < \alpha < 1$  and  $\{f_k\}$  is an arbitrary bounded sequence in  $\mathcal{B}^\alpha$  converging to 0 uniformly on the compact subsets of  $\mathbb{D}$  as  $k \rightarrow \infty$ , then we have that*

$$\lim_{k \rightarrow \infty} \sup_{z \in \mathbb{D}} |f_k(z)| = 0.$$

The following criterion for compactness follows from an easy modification of the Proposition 3.11 of [1]. Hence we omit the details.

**Lemma 2.6.** *Suppose  $X$  and  $Y$  are two Banach spaces. Then the weighted composition operator  $uC_\varphi : X \rightarrow Y$  is compact if whenever  $\{f_k\}$  is bounded in  $X$  and  $f_k \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$ , then  $uC_\varphi f_k \rightarrow 0$  in  $Y$  as  $k \rightarrow \infty$ .*

## 3. BOUNDEDNESS

In this section, we give some new characterizations for the boundedness of  $uC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$  in three cases.

**Theorem 3.1.** *If  $0 < \alpha < 1$ , then  $uC_\varphi$  maps  $\mathcal{B}^\alpha$  boundedly into  $\mathcal{Z}_\beta$  if and only if  $u \in \mathcal{Z}_\beta$  and*

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^\alpha} \asymp \sup_{n \geq 0} (n+1)^\alpha \|(2u'\varphi' + u\varphi'')\varphi^n\|_{\nu_\beta} < \infty. \quad (3.1)$$

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |u(z)\varphi'(z)^2|}{(1 - |\varphi(z)|^2)^{\alpha+1}} \asymp \sup_{n \geq 0} \|u(\varphi')^2 \varphi^n\|_{\nu_\beta} (n+1)^{\alpha+1} < \infty. \quad (3.2)$$

*Proof.* Sufficiency. Suppose  $u \in \mathcal{Z}_\beta$ , (3.1) and (3.2) hold. Since  $(uC_\varphi f)'' = u''C_\varphi f + (2u'\varphi' + u\varphi'')C_\varphi f' + u(\varphi')^2C_\varphi f''$ , using Lemma 2.3 and Lemma 2.4, for any  $f \in \mathcal{B}^\alpha$ ,

$$\begin{aligned} & (1 - |z|^2)^\beta |(uC_\varphi f)''(z)| \\ & \leq (1 - |z|^2)^\beta |u''(z)||f(\varphi(z))| + (1 - |z|^2)^\beta |u(z)(\varphi'(z))^2||f''(\varphi(z))| \\ & \quad + (1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)||f'(\varphi(z))| \\ & \preceq \|u\|_{\mathcal{Z}_\beta} \|f\|_{\mathcal{B}^\alpha} + \frac{(1 - |z|^2)^\beta |u(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^{\alpha+1}} \|f\|_{\mathcal{B}^\alpha} \\ & \quad + \frac{(1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^\alpha} \|f\|_{\mathcal{B}^\alpha} < \infty, \end{aligned}$$

and  $|(uC_\varphi f)'(0)| \preceq |u'(0)|\|f\|_{\mathcal{B}^\alpha} + \frac{|u(0)\varphi'(0)|}{(1 - |\varphi(0)|^2)^\alpha} \|f\|_{\mathcal{B}^\alpha}$ ,  $|uC_\varphi f(0)| \preceq \|f\|_{\mathcal{B}^\alpha}$ . From above it follows the boundedness of  $uC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$ .

Necessity. Suppose  $uC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$  is bounded for  $0 < \alpha < 1$ . Then choose the functions  $f(z) = 1$ ,  $f(z) = z$ ,  $f(z) = z^2$ , and for a fixed  $w \in \mathbb{D}$ , take

$$\begin{aligned} g_w(z) &= \frac{1 - |w|^2}{(1 - \bar{w}z)^\alpha} - \frac{\alpha(1 - |w|^2)^2}{(\alpha + 2)(1 - \bar{w}z)^{\alpha+1}} - \frac{2}{(\alpha + 2)(1 - |w|^2)^{\alpha-1}}, \\ f_w(z) &= \frac{1 - |w|^2}{(1 - \bar{w}z)^\alpha} - \frac{\alpha(1 - |w|^2)^2}{(\alpha + 1)(1 - \bar{w}z)^{\alpha+1}} - \frac{1}{(\alpha + 1)(1 - |w|^2)^{\alpha-1}}, \end{aligned}$$

Then it follows that  $u \in \mathcal{Z}_\beta$ , and

$$\begin{aligned} & \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)| < \infty, \\ & \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |u(z)\varphi'(z)^2| < \infty. \\ & \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |u(z)\varphi'(z)^2||\varphi(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+1}} < \infty, \\ & \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^\alpha} < \infty. \end{aligned}$$

From the above four inequalities, we obtain that

$$M_1 := \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1 - |\varphi(z)|^2)^\alpha} < \infty, \quad (3.3)$$

and

$$M_2 := \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |u(z) \varphi'(z)^2|}{(1 - |\varphi(z)|^2)^{\alpha+1}} < \infty. \quad (3.4)$$

Then employing Lemma 2.1 (i) with  $\nu = \nu_\alpha$ ,  $w(z) = \nu_\beta$ , Lemma 2.2 (i), (3.3) and (3.4), it follows that

$$\begin{aligned} M_1 &\asymp \sup_{n \geq 0} \frac{\|(2u'\varphi' + u\varphi'')\varphi^n\|_{\nu_\beta} (n+1)^\alpha}{\|z^n\|_{\nu_\alpha} (n+1)^\alpha} \\ &\asymp \sup_{n \geq 0} (n+1)^\alpha \|(2u'\varphi' + u\varphi'')\varphi^n\|_{\nu_\beta}, \\ M_2 &\asymp \sup_{n \geq 0} \frac{\|u(\varphi')^2 \varphi^n\|_{\nu_\beta} (n+1)^{\alpha+1}}{\|z^n\|_{\nu_{\alpha+1}} (n+1)^{\alpha+1}} \\ &\asymp \sup_{n \geq 0} \|u(\varphi')^2 \varphi^n\|_{\nu_\beta} (n+1)^{\alpha+1}. \end{aligned}$$

From the above inequality we obtain (3.1) and (3.2). This completes the proof of the theorem.  $\square$

**Theorem 3.2.** *If  $\alpha = 1$ , then  $uC_\varphi$  maps  $\mathcal{B}$  boundedly into  $\mathcal{Z}_\beta$  if and only if (3.1) and (3.2) hold and*

$$\sup_{n \geq 0} (\log n) \|u''(z) \varphi^n\|_{\nu_\beta} \asymp \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |u''(z)| \log \frac{e}{1 - |\varphi(z)|^2} < \infty. \quad (3.5)$$

*Proof.* Sufficiency. This part is similar to the proof in Theorem 3.1.

Necessity. Suppose that  $uC_\varphi$  maps  $\mathcal{B}$  boundedly into  $\mathcal{Z}_\beta$ . Similar to the proof in Theorem 3.1. we can obtain (3.1) and (3.2) with  $\alpha = 1$ , thus we need only to show (3.5). In this case, we choose the function

$$h_w(z) = \frac{6}{a} \left( \log \frac{2}{1 - \bar{w}z} \right)^2 - \frac{2}{a^2} \left( \log \frac{2}{1 - \bar{w}z} \right)^3,$$

where  $a = \log \frac{2}{1 - |w|^2}$ , then by (3.1) and (3.2), we can easily obtain that

$$M_3 := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |u''(z)| \log \frac{e}{1 - |\varphi(z)|^2} < \infty.$$

Then by Lemma 2.1 (i) with  $\nu = \left( \log \left( \frac{e}{1 - |z|^2} \right) \right)^{-1} = \nu_{\log}$ ,  $w(z) = \nu_\beta$ , and Lemma 2.2 (ii),

$$\begin{aligned} M_3 &= \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |u''(z)|}{\left( \log \frac{e}{1 - |\varphi(z)|^2} \right)^{-1}} \asymp \sup_{n \geq 0} \frac{\|u'' \varphi^n\|_{\nu_\beta} (\log n)}{\|z^n\|_{\nu_{\log}} (\log n)} \\ &\asymp \sup_{n \geq 0} (\log n) \|u'' \varphi^n\|_{\nu_\beta} < \infty. \end{aligned}$$

from which (3.5) follows. This completes the proof.  $\square$

**Theorem 3.3.** *If  $\alpha > 1$ , then  $uC_\varphi$  maps  $\mathcal{B}^\alpha$  boundedly into  $\mathcal{Z}_\beta$  if and only if (3.1) and (3.2) hold and*

$$\sup_{n \geq 0} (n+1)^{\alpha-1} \|u'' \varphi^n\|_{\nu_\beta} \asymp \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |u''(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} < \infty. \quad (3.6)$$

*Proof.* Sufficiency. This part is similar to the proof in Theorem 3.1.

Necessity. Take the function

$$Q_w(z) = \frac{(\alpha + 2)(1 - |w|^2)}{\alpha(1 - \bar{w}z)^\alpha} - \frac{(1 - |w|^2)^2}{(1 - \bar{w}z)^{\alpha+1}},$$

Employing the necessity in Theorem 3.1 with  $\alpha > 1$ , we can obtain that (3.1) and (3.2). Then we can easily obtain that

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |u''(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} < \infty.$$

Similar to showing the equivalence in (3.2) we obtain (3.6). This completes the proof.  $\square$

#### 4. ESSENTIAL NORMS

In this section we estimate the essential norms of  $uC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$  in terms of  $u, \varphi$ , their derivatives and  $\varphi^n$ . Denote  $\tilde{\mathcal{B}}^\alpha = \{f \in \mathcal{B}^\alpha : f(0) = 0\}$ . Let  $D_\alpha : \mathcal{B}^\alpha \rightarrow H_{\nu_\alpha}^\infty$  and  $S_\alpha : \mathcal{B}^\alpha \rightarrow H_{\nu_{\alpha+1}}^\infty$  be the first-order derivative operator and the second-order derivative operator, respectively. By Lemma 2.2 we have that

$$\|D_\alpha f\|_{H_{\nu_\alpha}} = \|f\|_{\mathcal{B}_\alpha} \text{ and } \|S_\alpha f\|_{H_{\nu_{\alpha+1}}} \asymp \|f\|_{\mathcal{B}_\alpha} \text{ for } f \in \tilde{\mathcal{B}}^\alpha.$$

Further by  $(uC_\varphi f)'' = u''C_\varphi f + (2u'\varphi' + u\varphi'')C_\varphi f + u(\varphi')^2C_\varphi f''$ , it follows that

$$\begin{aligned} \|uC_\varphi\|_{e, \tilde{\mathcal{B}}^\alpha \rightarrow \mathcal{Z}_\beta} &\preceq \|u''C_\varphi\|_{e, \tilde{\mathcal{B}}^\alpha \rightarrow H_{\nu_\beta}^\infty} + \|u(\varphi')^2C_\varphi\|_{e, H_{\nu_{\alpha+1}}^\infty \rightarrow H_{\nu_\beta}^\infty} \\ &\quad + \|(2u'\varphi' + u\varphi'')C_\varphi\|_{e, H_{\nu_\alpha}^\infty \rightarrow H_{\nu_\beta}^\infty}. \end{aligned} \quad (4.1)$$

For the upper bound, we only need to estimate the right three essential norms. It is obvious that every compact operator  $T \in \mathcal{K}(\tilde{\mathcal{B}}^\alpha, \mathcal{Z}_\beta)$  can be extended to a compact operator  $K \in \mathcal{K}(\mathcal{B}^\alpha, \mathcal{Z}_\beta)$ . In fact, for every  $f \in \mathcal{B}^\alpha$ ,  $f - f(0) \in \tilde{\mathcal{B}}^\alpha$ , and we can define  $K(f) := T(f - f(0)) + f(0)$ , which is a compact operator from  $\mathcal{B}^\alpha$  to  $\mathcal{Z}_\beta$ , due to  $K(f_k)$  has convergent subsequence when  $\{f_k\}$  is a bounded sequence. In the following lemma we consider the compact operator  $K_r$  on the space  $\mathcal{B}^\alpha$  defined by  $K_r f(z) = f(rz)$ .

**Lemma 4.1.** *If  $0 < \alpha < \infty$  and  $uC_\varphi$  is a bounded weighted composition operator from  $\mathcal{B}^\alpha$  into  $\mathcal{Z}_\beta$ , then*

$$\|uC_\varphi\|_{e, \tilde{\mathcal{B}}^\alpha \rightarrow \mathcal{Z}_\beta} = \|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta}.$$

*Proof.* Although the proof is similar to [2, Lemma 3.1], we give the process for the convenience of the readers. It is obvious that  $\|uC_\varphi\|_{e, \tilde{\mathcal{B}}^\alpha \rightarrow \mathcal{Z}_\beta} \leq \|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta}$ . For the converse, let  $T \in \mathcal{K}(\mathcal{B}^\alpha, \mathcal{Z}_\beta)$  be given. Choose an increasing sequence  $(r_n)$  in  $(0, 1)$  converging to 1. Denote  $\mathcal{A}$  the closed subspace of  $\mathcal{B}^\alpha$  consists of all constant functions. Then we have

$$\begin{aligned} \|uC_\varphi - T\|_{\mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} &= \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \|uC_\varphi(f) - T(f)\|_{\mathcal{Z}_\beta} \\ &\leq \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \|uC_\varphi(f - f(0)) - T|_{\tilde{\mathcal{B}}^\alpha}(f - f(0))\|_{\mathcal{Z}_\beta} + \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} \|uC_\varphi(f(0)) - T(f(0))\|_{\mathcal{Z}_\beta} \\ &\leq \sup_{g \in \tilde{\mathcal{B}}^\alpha} \|uC_\varphi(g) - T|_{\tilde{\mathcal{B}}^\alpha}(g)\|_{\mathcal{Z}_\beta} + \sup_{h \in \mathcal{A}} \|uC_\varphi(h) - T|_{\mathcal{A}}(h)\|_{\mathcal{Z}_\beta}. \end{aligned}$$

Hence

$$\begin{aligned}
\inf_{T \in \mathcal{K}(\mathcal{B}^\alpha, \mathcal{Z}_\beta)} \|uC_\varphi - T\|_{\mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} &\leq \inf_{T \in \mathcal{K}(\mathcal{B}^\alpha, \mathcal{Z}_\beta)} \|uC_\varphi - T\|_{\tilde{\mathcal{B}}^\alpha} \|\tilde{\mathcal{B}}^\alpha \rightarrow \mathcal{Z}_\beta \\
&+ \inf_{T \in \mathcal{K}(\mathcal{B}^\alpha, \mathcal{Z}_\beta)} \|uC_\varphi - T\|_{\mathcal{A}} \|\mathcal{A} \rightarrow \mathcal{Z}_\beta \\
&\leq \|uC_\varphi\|_{e, \tilde{\mathcal{B}}^\alpha \rightarrow \mathcal{Z}_\beta} + \lim_{n \rightarrow \infty} \|uC_\varphi(I - K_{r_n})\|_{\mathcal{A} \rightarrow \mathcal{Z}_\beta}.
\end{aligned}$$

Since  $uC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$  is bounded, it follows that

$$\lim_{n \rightarrow \infty} \|uC_\varphi(I - K_{r_n})\|_{\mathcal{A} \rightarrow \mathcal{Z}_\beta} \leq C \lim_{n \rightarrow \infty} \|I - K_{r_n}\|_{\mathcal{A} \rightarrow \mathcal{Z}_\beta} = 0.$$

Thus we obtain  $\|uC_\varphi\|_{e, \tilde{\mathcal{B}}^\alpha \rightarrow \mathcal{Z}_\beta} \geq \|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta}$ . The proof is finished.  $\square$

Since  $uC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$  is bounded, then  $u''C_\varphi$  maps  $\mathcal{B}^\alpha$  boundedly in  $H_{\nu_\beta}^\infty$  from  $u \in \mathcal{Z}_\beta$  for  $0 < \alpha < 1$ , (3.5) for  $\alpha = 1$  and (3.6) for  $\alpha > 1$ . Then from Lemma 4.1 we can get  $\|u''C_\varphi\|_{e, \tilde{\mathcal{B}}^\alpha \rightarrow H_{\nu_\beta}^\infty} = \|u''C_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty}$ . By (4.1) we have that

$$\begin{aligned}
\|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} &\preceq \|u''C_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty} + \|u(\varphi')^2 C_\varphi\|_{e, H_{\nu_{\alpha+1}}^\infty \rightarrow H_{\nu_\beta}^\infty} \\
&+ \|(2u'\varphi' + u\varphi'')C_\varphi\|_{e, H_{\nu_\alpha}^\infty \rightarrow H_{\nu_\beta}^\infty}.
\end{aligned} \tag{4.2}$$

In next lemma we give the estimates for the essential norm for  $uC_\varphi : \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty$ .

**Lemma 4.2.** *Let  $0 < \alpha < \infty$ , the weighted composition operator  $uC_\varphi : \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty$  be bounded.*

- (i) *If  $0 < \alpha < 1$ , then  $uC_\varphi : \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty$  is compact.*
- (ii) *If  $\alpha = 1$ , then*

$$\|uC_\varphi\|_{e, \mathcal{B} \rightarrow H_{\nu_\beta}^\infty} \asymp \limsup_{n \rightarrow \infty} (\log n) \|u\varphi^n\|_{\nu_\beta}.$$

- (iii) *If  $\alpha > 1$ , then*

$$\|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty} \asymp \limsup_{n \rightarrow \infty} (n+1)^{\alpha-1} \|u\varphi^n\|_{\nu_\beta}.$$

*Proof.* (i) Since  $uC_\varphi : \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty$  is bounded. Choose  $f(z) = 1$ , we can obtain  $u \in H_{\nu_\beta}^\infty$ . If  $(f_n)$  is a bounded sequence in  $\mathcal{B}^\alpha$  converging to zero uniformly on compact subsets of  $\mathbb{D}$ . By Lemma 2.5 we have that

$$\|uC_\varphi(f_n)\|_{\nu_\beta} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |u(z)| |f_n(\varphi(z))| \leq \|u\|_{\nu_\beta} \sup_{z \in \mathbb{D}} |f_n(z)| = 0.$$

By Lemma 2.6 it follows that  $uC_\varphi : \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty$  is compact.

- (ii) For  $\alpha = 1$ . By [14, Theorem 3.4] with  $n = 1$ , we obtain that

$$\|uC_\varphi\|_{e, \mathcal{B} \rightarrow H_{\nu_\beta}^\infty} \asymp \limsup_{|\varphi(z)| \rightarrow 1} (1 - |z|^2)^\beta |u(z)| \log \frac{1 + |\varphi(z)|}{1 - |\varphi(z)|}.$$

since the function  $\log \frac{1+x}{1-x} \asymp \log \frac{e}{1-x^2}$ ,  $x \in [0, 1)$ , we have

$$\|uC_\varphi\|_{e, \mathcal{B} \rightarrow H_{\nu_\beta}^\infty} \asymp \limsup_{|\varphi(z)| \rightarrow 1} (1 - |z|^2)^\beta |u(z)| \log \frac{e}{1 - |\varphi(z)|^2}.$$

By Lemma 2.1 (ii) and Lemma 2.2 (ii) it follows that

$$\|uC_\varphi\|_{e, \mathcal{B} \rightarrow H_{\nu_\beta}^\infty} \asymp \limsup_{n \rightarrow \infty} \frac{(\log n) \|u\varphi^n\|_{\nu_\beta}}{\|z^n\|_{\nu_{\log}} (\log n)} \asymp \limsup_{n \rightarrow \infty} (\log n) \|u\varphi^n\|_{\nu_\beta}.$$

(iii) For  $\alpha > 1$ . By [14, Theorem 3.2] with  $n = 1$ , it follows that

$$\|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty} \asymp \limsup_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |u(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}}.$$

Similarly by Lemma 2.1 (ii) and Lemma 2.2 (i) it follows (iii). This completes the proof.  $\square$

**Theorem 4.3.** *Let  $0 < \alpha < 1$ , the weighted composition operator  $uC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$  is bounded. Then*

$$\begin{aligned} \|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} \asymp \max \Big\{ & \limsup_{n \rightarrow \infty} (n+1)^\alpha \|(2u'\varphi' + u\varphi'')\varphi^n\|_{\nu_\beta}, \\ & \limsup_{n \rightarrow \infty} (n+1)^{\alpha+1} \|u(\varphi')^2\varphi^n\|_{\nu_\beta} \Big\}. \end{aligned}$$

*Proof.* The boundedness of  $uC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$  implies that  $u''C_\varphi : \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty$ ,  $(2u'\varphi' + u\varphi'')C_\varphi : H_{\nu_\alpha}^\infty \rightarrow H_\beta^\infty$  and  $u(\varphi')^2C_\varphi : H_{\nu_{\alpha+1}}^\infty \rightarrow H_{\nu_\beta}^\infty$  are bounded weighted composition operators by Theorem 3.1.

The upper estimate. From Lemma 4.2 it follows that  $\|u''C_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty} = 0$ . On the other hand, by Lemma 2.1 (i) and Lemma 2.2 (i),

$$\begin{aligned} \|(2u'\varphi' + u\varphi'')C_\varphi\|_{e, H_{\nu_\alpha}^\infty \rightarrow H_{\nu_\beta}^\infty} &= \limsup_{n \rightarrow \infty} \frac{\|(2u'\varphi' + u\varphi'')\varphi^n\|_{\nu_\beta}}{\|z^n\|_{\nu_\alpha}} \\ &\asymp \limsup_{n \rightarrow \infty} (n+1)^\alpha \|(2u'\varphi' + u\varphi'')\varphi^n\|_{\nu_\beta}. \end{aligned}$$

$$\|u(\varphi')^2C_\varphi\|_{e, H_{\nu_{\alpha+1}}^\infty \rightarrow H_{\nu_\beta}^\infty} \asymp \limsup_{n \rightarrow \infty} (n+1)^{\alpha+1} \|u(\varphi')^2\varphi^n\|_{\nu_\beta}.$$

Thus by (4.2) we obtain that

$$\begin{aligned} \|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} \preceq \max \Big\{ & \limsup_{n \rightarrow \infty} (n+1)^\alpha \|(2u'\varphi' + u\varphi'')\varphi^n\|_{\nu_\beta}, \\ & \limsup_{n \rightarrow \infty} (n+1)^{\alpha+1} \|u(\varphi')^2\varphi^n\|_{\nu_\beta} \Big\}. \end{aligned}$$

The lower estimate. Let  $\{z_k\}$  be a sequence in  $\mathbb{D}$  such that  $|\varphi(z_k)| \rightarrow 1$  as  $n \rightarrow \infty$ . Define

$$f_k(z) = \frac{1 - |\varphi(z_k)|^2}{(1 - \overline{\varphi(z_k)}z)^\alpha} - \frac{\alpha(1 - |\varphi(z_k)|^2)^2}{(\alpha+1)(1 - \overline{\varphi(z_k)}z)^{\alpha+1}} - \frac{1}{(\alpha+1)(1 - |\varphi(z_k)|^2)^{\alpha-1}},$$

$$g_k(z) = \frac{1 - |\varphi(z_k)|^2}{(1 - \overline{\varphi(z_k)}z)^\alpha} - \frac{\alpha(1 - |\varphi(z_k)|^2)^2}{(\alpha+2)(1 - \overline{\varphi(z_k)}z)^{\alpha+1}} - \frac{2}{(\alpha+2)(1 - |\varphi(z_k)|^2)^{\alpha-1}},$$

It is obvious that  $f_k$  and  $g_k$  are bounded sequences in  $\mathcal{B}^\alpha$  and converge to zero uniformly on compact subset of  $\mathbb{D}$ . By Lemma 2.6, for every compact operator



$T : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$ , we have that  $\|Tf_k\|_{\mathcal{Z}_\beta} \rightarrow 0$  and  $\|Tg_k\|_{\mathcal{Z}_\beta} \rightarrow 0$  as  $k \rightarrow \infty$ . Thus

$$\begin{aligned}
\|uC_\varphi - T\|_{\mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} &\succeq \limsup_{k \rightarrow \infty} \|uC_\varphi(f_k)\|_{\mathcal{Z}_\beta} \\
&\succeq \limsup_{k \rightarrow \infty} \frac{\alpha(1 - |z_k|^2)^\beta |u(z_k)\varphi'(z_k)^2| |\varphi(z_k)|^2}{(1 - |\varphi(z_k)|^2)^{\alpha+1}} \\
&\succeq \limsup_{k \rightarrow \infty} \frac{(1 - |z_k|^2)^\beta |u(z_k)\varphi'(z_k)^2|}{(1 - |\varphi(z_k)|^2)^{\alpha+1}} \\
&\asymp \limsup_{n \rightarrow \infty} (n+1)^{\alpha+1} \|u(\varphi')^2 \varphi^n\|_{\nu_\beta}. \\
\|uC_\varphi - T\|_{\mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} &\preceq \limsup_{k \rightarrow \infty} \|uC_\varphi(g_k)\|_{\mathcal{Z}_\beta} \\
&\preceq \limsup_{k \rightarrow \infty} \frac{(1 - |z_k|^2)^\beta |2u'(z_k)\varphi'(z_k) + u(z_k)\varphi''(z_k)|}{(1 - |\varphi(z_k)|^2)^\alpha} \\
&\asymp \limsup_{n \rightarrow \infty} (n+1)^\alpha \|(2u'\varphi' + u\varphi'')\varphi^n\|_{\nu_\beta}.
\end{aligned}$$

From the above two inequalities we obtain the lower estimate. This completes the proof.  $\square$

In the next two theorems, we need the following test functions sequences. Let  $\{z_k\}$  be a sequence in  $\mathbb{D}$  such that  $|\varphi(z_k)| \rightarrow 1$  as  $k \rightarrow \infty$ . Define

$$f_k(z) = \frac{1 - |\varphi(z_k)|^2}{(1 - \overline{\varphi(z_k)}z)^\alpha} - \frac{(1 - |\varphi(z_k)|^2)^2}{(1 - \overline{\varphi(z_k)}z)^{\alpha+1}}, \quad (4.3)$$

$$g_k(z) = \frac{1 - |\varphi(z_k)|^2}{(1 - \overline{\varphi(z_k)}z)^\alpha} - \frac{\alpha(1 - |\varphi(z_k)|^2)^2}{(\alpha+1)(1 - \overline{\varphi(z_k)}z)^{\alpha+1}}, \quad (4.4)$$

$$h_k(z) = \frac{1 - |\varphi(z_k)|^2}{(1 - \overline{\varphi(z_k)}z)^\alpha} - \frac{\alpha(1 - |\varphi(z_k)|^2)^2}{(\alpha+2)(1 - \overline{\varphi(z_k)}z)^{\alpha+1}}. \quad (4.5)$$

It is easy to see that  $f_k$ ,  $g_k$ , and  $h_k$  are all in  $\mathcal{B}^\alpha$  and converge to zero uniformly on the compact subset of  $\mathbb{D}$ . Moreover,

$$f_k(\varphi(z_k)) = 0, \quad f'_k(\varphi(z_k)) = \frac{-\overline{\varphi(z_k)}}{(1 - |\varphi(z_k)|^2)^\alpha}, \quad f''_k(\varphi(z_k)) = \frac{-2(\alpha+1)\overline{\varphi(z_k)}}{(1 - |\varphi(z_k)|^2)^{\alpha+1}}.$$

$$g_k(\varphi(z_k)) = \frac{1}{(\alpha+1)(1 - |\varphi(z_k)|^2)^{\alpha-1}}, \quad g'_k(\varphi(z_k)) = 0, \quad g''_k(\varphi(z_k)) = \frac{-\alpha\overline{\varphi(z_k)}}{(1 - |\varphi(z_k)|^2)^{\alpha+1}}.$$

$$h_k(\varphi(z_k)) = \frac{2}{(\alpha+2)(1 - |\varphi(z_k)|^2)^{\alpha-1}}, \quad h'_k(\varphi(z_k)) = \frac{\alpha\overline{\varphi(z_k)}}{(\alpha+2)(1 - |\varphi(z_k)|^2)^\alpha}, \quad h''_k(\varphi(z_k)) = 0.$$

**Theorem 4.4.** *Let  $\alpha > 1$ , the weighted composition operator  $uC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$  is bounded. Then*

$$\begin{aligned}
\|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} &\asymp \max \left\{ \limsup_{n \rightarrow \infty} (n+1)^\alpha \|(2u'\varphi' + u\varphi'')\varphi^n\|_{\nu_\beta}, \right. \\
&\quad \limsup_{n \rightarrow \infty} (n+1)^{\alpha+1} \|u(\varphi')^2 \varphi^n\|_{\nu_\beta}, \\
&\quad \left. \limsup_{n \rightarrow \infty} (n+1)^{\alpha-1} \|u''\varphi^n\|_{\nu_\beta} \right\}.
\end{aligned}$$

*Proof.* The upper estimate. By Lemma 4.2 (iii) we obtain that

$$\|u''C_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow H_{\nu_\beta}^\infty} \asymp \limsup_{n \rightarrow \infty} (n+1)^{\alpha-1} \|u''\varphi^n\|_{\nu_\beta}.$$

Then by (4.2) and the proof for the upper estimate in Theorem 4.3, we obtain the upper estimate.

The lower estimate. Let  $\{z_k\}$  be a sequence in  $\mathbb{D}$  such that  $|\varphi(z_k)| \rightarrow 1$  as  $k \rightarrow \infty$ . Using test functions defined in (4.3)-(4.5) and for every compact operator  $T : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$ , it follows that

$$\begin{aligned} \|uC_\varphi - T\|_{\mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} &\geq \limsup_{k \rightarrow \infty} \|uC_\varphi(f_k)\|_{\mathcal{Z}_\beta} \\ &\geq \limsup_{k \rightarrow \infty} (1 - |z_k|^2)^\beta \left| (2u'(z_k)\varphi'(z_k) + u(z_k)\varphi''(z_k)) \frac{-\overline{\varphi(z_k)}}{(1 - |\varphi(z_k)|^2)^\alpha} \right. \\ &\quad \left. + u(z_k)(\varphi'(z_k))^2 \frac{-2(\alpha+1)\overline{(\varphi(z_k))^2}}{(1 - |\varphi(z_k)|^2)^{\alpha+1}} \right|. \end{aligned} \quad (4.6)$$

$$\begin{aligned} \|uC_\varphi - T\|_{\mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} &\geq \limsup_{k \rightarrow \infty} \|uC_\varphi(g_k)\|_{\mathcal{Z}_\beta} \\ &\geq \limsup_{k \rightarrow \infty} (1 - |z_k|^2)^\beta \left| u''(z_k) \frac{1}{(\alpha+1)(1 - |\varphi(z_k)|^2)^{\alpha-1}} \right. \\ &\quad \left. + u(z_k)(\varphi'(z_k))^2 \frac{-\alpha\overline{(\varphi(z_k))^2}}{(1 - |\varphi(z_k)|^2)^{\alpha+1}} \right|. \end{aligned} \quad (4.7)$$

$$\begin{aligned} \|uC_\varphi - T\|_{\mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} &\geq \limsup_{k \rightarrow \infty} \|uC_\varphi(h_k)\|_{\mathcal{Z}_\beta} \\ &\geq \limsup_{k \rightarrow \infty} (1 - |z_k|^2)^\beta \left| u''(z_k) \frac{2}{(\alpha+2)(1 - |\varphi(z_k)|^2)^{\alpha-1}} \right. \\ &\quad \left. + (2u'(z_k)\varphi'(z_k) + u(z_k)\varphi''(z_k)) \frac{\alpha\overline{\varphi(z_k)}}{(\alpha+2)(1 - |\varphi(z_k)|^2)^\alpha} \right|. \end{aligned} \quad (4.8)$$

Denote

$$\begin{aligned} A_k &:= \frac{(1 - |z_k|^2)^\beta u''(z_k)}{(1 - |\varphi(z_k)|^2)^{\alpha-1}}, \\ B_k &:= \frac{(1 - |z_k|^2)^\beta (2u'(z_k)\varphi'(z_k) + u(z_k)\varphi''(z_k))\overline{\varphi(z_k)}}{(1 - |\varphi(z_k)|^2)^\alpha}, \\ C_k &:= \frac{(1 - |z_k|^2)^\beta u(z_k)(\varphi'(z_k))^2 \overline{(\varphi(z_k))^2}}{(1 - |\varphi(z_k)|^2)^{\alpha+1}}. \end{aligned}$$

Then (4.6)-(4.8) become

$$\begin{aligned} \limsup_{k \rightarrow \infty} \|uC_\varphi(f_k)\|_{\mathcal{Z}_\beta} &\geq \limsup_{k \rightarrow \infty} |B_k + 2(\alpha+1)C_k|, \\ \limsup_{k \rightarrow \infty} \|uC_\varphi(g_k)\|_{\mathcal{Z}_\beta} &\geq \limsup_{k \rightarrow \infty} \left| \frac{A_k}{\alpha+1} - \alpha C_k \right|, \\ \limsup_{k \rightarrow \infty} \|uC_\varphi(h_k)\|_{\mathcal{Z}_\beta} &\geq \limsup_{k \rightarrow \infty} \left| \frac{2A_k}{\alpha+2} + \frac{\alpha}{\alpha+2} B_k \right|. \end{aligned}$$

By the boundedness of  $uC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$ , it follows that  $\limsup_{k \rightarrow \infty} \|uC_\varphi(f_k)\|_{\mathcal{Z}_\beta} < \infty$ ,  $\limsup_{k \rightarrow \infty} \|uC_\varphi(g_k)\|_{\mathcal{Z}_\beta} < \infty$  and  $\limsup_{k \rightarrow \infty} \|uC_\varphi(h_k)\|_{\mathcal{Z}_\beta} < \infty$ . Thus we can obtain

that  $\limsup_{k \rightarrow \infty} |A_k| < \infty$ ,  $\limsup_{k \rightarrow \infty} |B_k| < \infty$  and  $\limsup_{k \rightarrow \infty} |C_k| < \infty$ . Then by (4.6)-(4.8) it follows that

$$\begin{aligned} \|uC_\varphi\|_{e, \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} &= \inf_{T \in K(\mathcal{B}^\alpha, \mathcal{Z}_\beta)} \|uC_\varphi - T\|_{\mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} \\ &\succeq \max \left\{ \limsup_{|\varphi(z_k)| \rightarrow 1} \frac{(1 - |z_k|^2)^\beta |u''(z_k)|}{(1 - |\varphi(z_k)|^2)^{\alpha-1}}, \right. \\ &\quad \limsup_{|\varphi(z_k)| \rightarrow 1} \frac{(1 - |z_k|^2)^\beta |2u'(z_k)\varphi'(z_k) + u(z_k)\varphi''(z_k)|}{(1 - |\varphi(z_k)|^2)^\alpha}, \\ &\quad \left. \limsup_{|\varphi(z_k)| \rightarrow 1} \frac{(1 - |z_k|^2)^\beta |u(z_k)(\varphi'(z_k))^2|}{(1 - |\varphi(z_k)|^2)^{\alpha+1}} \right\}. \end{aligned}$$

Then using Lemma 2.1(ii) and Lemma 2.2(i) it follows the lower estimate. This completes the proof.  $\square$

**Theorem 4.5.** *Weighted composition operator  $uC_\varphi : \mathcal{B} \rightarrow \mathcal{Z}_\beta$  is bounded. Then*

$$\begin{aligned} \|uC_\varphi\|_{e, \mathcal{B} \rightarrow \mathcal{Z}_\beta} &\asymp \max \left\{ \limsup_{n \rightarrow \infty} (n+1) \|(2u'\varphi' + u\varphi'')\varphi^n\|_{\nu_\beta}, \right. \\ &\quad \limsup_{n \rightarrow \infty} (n+1)^2 \|u(\varphi')^2\varphi^n\|_{\nu_\beta}, \\ &\quad \left. \limsup_{n \rightarrow \infty} (\log n) \|u''\varphi^n\|_{\nu_\beta} \right\}. \end{aligned}$$

*Proof.* By Lemma 4.2 (ii), we obtain that

$$\|u''C_\varphi\|_{e, \mathcal{B} \rightarrow H_{\nu_\beta}^\infty} \asymp \limsup_{n \rightarrow \infty} (\log n) \|u''\varphi^n\|_{\nu_\beta}.$$

Then by (4.2) and the proof for the upper estimate in Theorem 4.3, we obtain the upper estimate.

The lower estimate. Similar to the proof in Theorem 4.4, we obtain (4.6)-(4.8) with  $\alpha = 1$ . Then

$$\begin{aligned} \|uC_\varphi T\|_{e, \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta} &\succeq \max \left\{ \limsup_{k \rightarrow \infty} (1 - |z_k|^2)^\beta |u''(z_k)|, \right. \\ &\quad \limsup_{k \rightarrow \infty} \frac{(1 - |z_k|^2)^\beta |2u'(z_k)\varphi'(z_k) + u(z_k)\varphi''(z_k)|}{1 - |\varphi(z_k)|^2}, \\ &\quad \left. \limsup_{k \rightarrow \infty} \frac{(1 - |z_k|^2)^\beta |u(z_k)(\varphi'(z_k))^2|}{(1 - |\varphi(z_k)|^2)^2} \right\}. \end{aligned} \quad (4.9)$$

By Lemma 2.1 (ii), Lemma 2.2 (i) and (4.9) we have that

$$\begin{aligned} \|uC_\varphi\|_{e, \mathcal{B} \rightarrow \mathcal{Z}_\beta} &\asymp \max \left\{ \limsup_{n \rightarrow \infty} (n+1) \|(2u'\varphi' + u\varphi'')\varphi^n\|_{\nu_\beta}, \right. \\ &\quad \left. \limsup_{n \rightarrow \infty} (n+1)^2 \|u(\varphi')^2\varphi^n\|_{\nu_\beta} \right\}. \end{aligned}$$

Now choose the function

$$\tilde{h}_k(z) = \frac{6}{a_k} \left( \log \frac{2}{1 - \overline{\varphi(z_k)}z} \right)^2 - \frac{2}{a_k^2} \left( \log \frac{2}{1 - \overline{\varphi(z_k)}z} \right)^3,$$

where  $a_k = \log \frac{2}{1-|\varphi(z_k)|^2}$ . It is easy to find that the sequence  $\{\tilde{h}_k\} \in \mathcal{B}$  converges to zero on the compact subset of  $\mathbb{D}$ , then we can obtain

$$\limsup_{k \rightarrow \infty} (1 - |z_k|^2)^\beta |u''(z_k)| \log \frac{e}{1 - |\varphi(z_k)|^2} \leq \|uC_\varphi\|_{e, \mathcal{B} \rightarrow \mathcal{Z}_\beta}.$$

Further employing Lemma 2.1 (ii) and Lemma 2.2 (ii), we get that

$$\|uC_\varphi\|_{e, \mathcal{B} \rightarrow \mathcal{Z}_\beta} \asymp \limsup_{n \rightarrow \infty} (\log n) \|u''\varphi^n\|_{\nu_\beta}.$$

This completes the proof.  $\square$

From the above Theorems, we obtain the new equivalent conditions for the compactness of  $uC_\varphi : \mathcal{B} \rightarrow \mathcal{Z}_\beta$ .

**Corollary 4.6.** *Suppose that  $u \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$ , then the followings are equivalent:*

- (a)  $uC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$  is compact.
- (b)  $uC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{Z}_\beta$  is bounded and
- (i) For  $0 < \alpha < 1$ ,

$$\limsup_{n \rightarrow \infty} (n+1)^\alpha \|(2u'\varphi' + u\varphi'')\varphi^n\|_{\nu_\beta} = 0, \quad (4.10)$$

$$\limsup_{n \rightarrow \infty} (n+1)^{\alpha+1} \|u(\varphi')^2\varphi^n\|_{\nu_\beta} = 0. \quad (4.11)$$

- (ii) For  $\alpha = 1$ , (4.10), (4.11) hold and  $\limsup_{n \rightarrow \infty} (\log n) \|u''\varphi^n\|_{\nu_\beta} = 0$ .
- (iii) For  $\alpha > 1$ , (4.10), (4.11) hold and  $\limsup_{n \rightarrow \infty} (n+1)^{\alpha-1} \|u''\varphi^n\|_{\nu_\beta} = 0$ .

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